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Report TW 69

# On the diffraction by a slit or a ribbon

by

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February 1961

## 1. Introduction

In a previous report [1] the solution of the two-dimensional diffraction problem for a half-plane was obtained by means of a Green's function for a Riemannian plane of two sheets with one branch-point. In the present paper the solution of the two-dimensional diffraction problems for a slit and a ribbon will be obtained by means of a Green's function for a Riemannian plane of two sheets with two branch-points. A series expansion of this Green's function in terms of Mathieufunction is given. Application of the method of images in the Riemannian plane then leads to solutions of the diffraction problems, which have been already obtained by others, either in the case of a cylindrical incident wave , or in its limiting case of a plane incident wave, [3] , [4] and [5]. The present paper, hence, gives an alternative method to obtain known results.

## 2. General considerations

Diffraction is governed by the Helmholtz equation

$$\Delta \varphi + k^2\varphi = 0, \tag{2.1}$$

which is obtained from the wave equation $\Delta \psi - c^{-2}\partial^2 \psi / \partial t^2 = 0$ by putting $\psi = \exp(i\omega t)$, $k = \omega/c$.

Solutions of (2.1) will be required to satisfy Sommerfeld's radiation condition at infinity, viz.

$$r\left(\frac{\partial \varphi}{\partial r} - ik\varphi\right) \to 0 \quad \text{for} \quad r \to \infty \tag{2.2}$$

for every direction in which the region, in which $\varphi$ is defined, extends to infinity. At the boundaries of this region $\varphi$ will either be required to satisfy homogeneous Dirichlet or Neumann conditions.

The Green's function associated with (2.1), valid for the full x-y plane and with its logarithmic singularity in $(x_o, y_o)$ satisfies

$$(\Delta + k^2)G_o(x,y,x_o,y_o) = -\delta(x-x_o)\delta(y-y_o), \tag{2.3}$$

and also the radiation condition (2.2). This function is known to be a Hankelfunction of the first kind, viz.

$$G_o(x,y,x_o,y_o) = \tfrac{1}{4}i\, H_o^{(1)}\left(k\sqrt{(x-x_o)^2+(y-y_o)^2}\right). \tag{2.4}$$

Introduction of elliptic coordinates

$$x = a\cosh\mu\,\cos\vartheta\ , \quad y = a\sinh\mu\,\sin\vartheta\ , \tag{2.5}$$

transforms (2.1) in

$$\frac{\partial^2 \varphi}{\partial \mu^2} + \frac{\partial^2 \varphi}{\partial \vartheta^2} + 2h^2(\cosh 2\mu - \cos 2\vartheta) = 0, \tag{2.6}$$

with

$$h = \tfrac{1}{2}ak. \tag{2.7}$$

The mapping achieved by (2.5) depends on the ranges allowed for the variables $\mu$ and $\vartheta$. For $0 \leq \mu < \infty$, $-\pi < \vartheta \leq \pi$ we have a one to one mapping of the full x-y plane on a semi-infinite rectangle in the $\mu - \vartheta$ plane. The radiation condition (2.2) then assumes the form

$$\frac{\partial \varphi}{\partial \mu} - ihe^{\mu}\varphi \longrightarrow 0 \quad \text{for } \mu \longrightarrow \infty . \tag{2.8}$$

For $-\infty < \mu < \infty$, $-\pi < \vartheta \leq \pi$ every point in the x-y plane correspond to two points in the $\mu - \vartheta$ plane. The transformation (2.5) then can be interpreted as the mapping of a Riemannian x-y plane of two sheets and branch-points in $(\pm a, 0)$ on the strip $-\pi < \vartheta \leq \pi$ of the $\mu - \vartheta$ plane, or also on the mantle of a circular cylinder with unit radius and with axial coordinate $\mu$ and polar coordinate $\vartheta$. If the first sheet is taken to correspond with the upper half $\mu > 0$ of the cylinder-mantle, and the second sheet with its lower half $\mu < 0$, the radiation condition again is found to have the form (2.8), since it is only postulated for the first sheet. However, if the first sheet is taken to correspond with the halfmantle $0 < \vartheta < \pi$ and the second sheet with the halfmantle $-\pi < \vartheta < 0$, the radiation condition (2.2) reads

$$\frac{\partial \varphi}{\partial \mu} - ihe^{\mu}\varphi \longrightarrow 0 \quad \text{for } \mu \longrightarrow \pm\infty . \tag{2.8'}$$

In order to obtain a functions in the Riemannian plane which satisfy the radiation condition in the first sheet when it is not known before hand with which part of the cylindermantle this first sheet correspond we should clearly require that such a function satisfies (2.8') as far as its region extends to infinity.

The Green's function associated with (2.6), valid for the full cylinder mantle and with its singularity in $(\mu_0, \vartheta_0)$ satisfies (2.8') and

$$\left[\frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial \vartheta^2} + 2h^2(\cosh 2\mu - \cos 2\vartheta)\right] G(\mu, \vartheta, \mu_0, \vartheta_0) = \\ = -\delta(\mu - \mu_0)\delta(\vartheta - \vartheta_0). \tag{2.9}$$

Although the Green's functions $G_0(x, y, x_0, y_0)$ and $G(\mu, \vartheta, \mu_0, \vartheta_0)$ have the same singularity at corresponding points, they are not identical. Indeed the region in which $G_0(x,y,x_0,y_0)$ is valid covers the first sheet of the Riemannian plane, and the region in which $G(\mu,\vartheta,\mu_0,\vartheta_0)$ is valid both sheets. In particular $G_0(x,y,x_0,y_0)$ is continuous along any path in the first sheet which crosses the segment $-a<x<a$, $y=0$, but $G(\mu,\vartheta,\mu_0,\vartheta_0)$ shows a jump along such a path, if, for the sake of definiteness, the first sheet is taken to correspond with the upper half $\mu>0$ of the cylinder mantle.

Now consider the combination $G(\mu,\vartheta,\mu_0,\vartheta_0)+G(\mu,\vartheta,-\mu_0,-\vartheta_0$ Because of the relation $G(\mu,\vartheta,-\mu_0,-\vartheta_0)=G(-\mu,-\vartheta,\mu_0,\vartheta_0)$ which follows from the symmetry properties of the differential operator $\partial^2/\partial\mu^2+\partial^2/\partial\vartheta^2+2h^2(\cosh 2\mu-\cos 2\vartheta)$ (comp.[1]), this combination again is continuous along any path in the first sheet which crosses the segment $-a<x<a$, $y=0$. In fact it is regular in the first sheet, with the exception of the point $(\mu_0,\vartheta_0)$. Since it also satisfies the radiation condition it must be identical in the first sheet with $G_0(x,y,x_0,y_0)$. In the second sheets it provides a continuation of $G_0(x,y,x_0,y_0)$ which initially was undefined there. We hence may put

$$G_0(x,y,x_0,y_0)=G(\mu,\vartheta,\mu_0,\vartheta_0)+G(\mu,\vartheta,-\mu_0,-\vartheta_0)\ . \qquad (2.10)$$

This relation is aequivalent with the following transformation formula for the two-dimensional delta-function

$$\delta(x-x_0)\,\delta(y-y_0)= \qquad (2.11)$$
$$=\tfrac{1}{2}a^2(\cosh 2\mu-\cos 2\vartheta)\left[\delta(\mu-\mu_0)\,\delta(\vartheta-\vartheta_0)+\delta(\mu+\mu_0)\,\delta(\vartheta+\vartheta_0)\right].$$

By means of (2.5) and (2.11), (2.3) indeed is transformed in

$$\left[\frac{\partial^2}{\partial\mu^2}+\frac{\partial^2}{\partial\vartheta^2}+2h^2(\cosh 2\mu-\cos 2\vartheta)\right]G(x,y,x_0,y_0)=$$
$$=-\delta(\mu-\mu_0)\,\delta(\vartheta-\vartheta_0)-\delta(\mu+\mu_0)\,\delta(\vartheta+\vartheta_0).$$

Once the function $G(\mu,\vartheta,\mu_0,\vartheta_0)$ is known we can easily construct solutions for the diffraction problem with Dirichlet or Neumann conditions on a ribbon or on the screen outside a slit. Consider incident cylindrical waves with source in $(x_0,y_0)$. These are represent by the Green's function $G_0(x,y,x_0,y_0)$ which we will denote in the remaining part of this section by $G_0$.

Now consider the function

$$G_{Dr}=G(\mu,\vartheta,\mu_0,\vartheta_0)-G(\mu,\vartheta,-\mu_0,\vartheta_0). \qquad (2.12)$$

Clearly $G_{Dr}=0$ for $\mu=0$ because of the symmetry relation $G(\mu,\vartheta,-\mu_o,\vartheta_o) = G(-\mu,\vartheta,\mu_o,\vartheta_o)$. Moreover it has only one source in the first sheet, which corresponds to the source of the incident waves and it satisfies the radiation condition at infinity because this condition is linear and homogeneous. It hence represents the solution of the diffraction problem for a ribbon $\mu=0$ or $-a<x<a$, $y=0$ with Dirichlet conditions on the ribbon. In an analogous way we find that

$$G_{Nr} = G(\mu,\vartheta,\mu_o,\vartheta_o) + G(\mu,\vartheta,-\mu_o,\vartheta_o) \qquad (2.13)$$

is the solution of the diffraction problem for a ribbon with Neumann conditions on the ribbon. The scattered waves for these two problems are respectively given by

$$G^*_{Dr} = G_o-G_{Dr}=-G(\mu,\vartheta,-\mu_o,\vartheta_o) - G(\mu,\vartheta,-\mu_o,-\vartheta_o), \quad (2.12')$$

and

$$G^*_{Nr} = G_o-G_{Nr} = G(\mu,\vartheta,-\mu_o,\vartheta_o) - G(\mu,\vartheta,-\mu_o,-\vartheta_o). \quad (2.13')$$

In the case of a slit we think the first sheet of the Riemannian plane delimited from the second sheet by the screen, i.e. by the lines $\vartheta=0$ and $\vartheta=\pi$, or $x<-a$, $y=0$ and $a<x$, $y=0$. In fact it appears that the reasoning that led to (2.10) essentially is independent of the manner in which we delimit both sheets of the Riemannian plane. In the present context we only require that $(\mu_o,\vartheta_o)$ lies in the first sheet. By a reasoning analogous to that given before we find then that

$$G_{Ds} = G(\mu,\vartheta,\mu_o,\vartheta_o) - G(\mu,\vartheta,\mu_o,-\vartheta_o) \qquad (2.14)$$

is the solution of the diffraction problem for a slit $-a<x<a$, $y=0$ or $\mu=0$ in a screen $x<-a$, $y=0$ and $a<x$, $y=0$, or $\vartheta=0$ and $\vartheta=\pi$ with Dirichlet conditions on the screen, and that

$$G_{Ns} = G(\mu,\vartheta,\mu_o,\vartheta_o) + G(\mu,\vartheta,\mu_o,-\vartheta_o)$$

is the solution of the diffraction problem for this slit with Neumann conditions on the screen. The scattered waves for the latter two problems are given respectively by

$$G^*_{Ds} = G_o-G_{Ds} = -G(\mu,\vartheta,\mu_o,-\vartheta_o)-G(\mu,\vartheta,-\mu_o,-\vartheta_o), \qquad (2.14')$$

and

$$G^*_{Ns} = G_o-G_{Ns} = G(\mu,\vartheta,\mu_o,-\vartheta_o) + G(\mu,\vartheta,-\mu_o,-\vartheta_o). \qquad (2.15')$$

From the formulae (2.12), (2.13), (2.14) and (2.15) follows the extended Babinet's principle in the form

$$G_{Ns}-G_{Dr}=G_{Nr}-G_{Ds}=G(\mu,\vartheta,-\mu_o,\vartheta_o)+G(\mu,\vartheta,\mu_o,-\vartheta_o), \qquad (2.16)$$

or also

$$G^*_{Dr}-G^*_{Ns}=G^*_{Ds}-G^*_{Ns}=G(\mu,\vartheta,-\mu_o,\vartheta_o)+G(\mu,\vartheta,\mu_o,-\vartheta_o). \qquad (2.16')$$

3. Determination of $G(\mu,\vartheta,\mu_o,\vartheta_o)$

We will now construct the solution of

$$\frac{\partial^2 G}{\partial\mu^2}+\frac{\partial^2 G}{\partial\vartheta^2}+2h^2(\cosh 2\mu-\cos 2\vartheta)G=-\delta(\mu-\mu_o)\delta(\vartheta-\vartheta_o),$$

$$(-\infty<\mu<\infty,\ -\pi<\vartheta\leq\pi), \qquad (3.1)$$

under the conditions

$$\frac{\partial G}{\partial\mu}-ihe^{\mu}G\longrightarrow 0 \qquad \text{for } \mu\longrightarrow\pm\infty. \qquad (3.2)$$

The homogeneous equation corresponding to (3.1) is separable. Putting $G=M(\mu)\Theta(\vartheta)$ we find the two Mathieu equations

$$M''-(\lambda-2h^2\cosh 2\mu)M=0, \qquad (3.3)$$

and

$$\Theta''+(\lambda-2h^2\cos 2\vartheta)\Theta=0. \qquad (3.4)$$

In the preceding section we have restricted $\vartheta$ to values in the interval $-\pi<\vartheta\leq\pi$. A more basic point of view is at first to lay down no restrictions on $\vartheta$, but to require, that $(\mu,\vartheta\pm 2n\pi)$, $n=0,1,2,\ldots$ are aequivalent representations of the same point in the Riemannian plane, or in other words, that any function in the $\mu-\vartheta$ plane should have a periodicity $2\pi$ in $\vartheta$. The original restriction on $\vartheta$, then, is only a question of economizing. It also becomes self-evident that the Riemannian plane can be mapped on the mantle of circular cylinder, as was stated before. From the above consideration it follows that we have to deal with the periodic solutions of (3.4) with period $2\pi$ and $\pi$, i.e. the Mathieufunctions $ce_n(\vartheta,h^2)$, $n=0,1,2..$ and $se_n(\vartheta,h^2)$, $n=1,2,3,\ldots$ . The parameter $h^2$ will be omitted in the sequel.

Returning to the original inhomogeneous equation (3.1) we try a series solution in terms of Mathieufunctions and put

$$G = \sum_{n=0}^{\infty} \varepsilon_n \left[ A_n(\mu) ce_n(\vartheta) ce_n(\vartheta_o) + B_n(\mu) se_n(\vartheta) se_n(\vartheta_o) \right], \quad (3.5)$$

where the factors $ce_n(\vartheta_o)$ and $se_n(\vartheta_o)$ have been added at the outset because of the known invariance of Green's functions for interchanging $\vartheta$ with $\vartheta_o$ and $\mu$ with $\mu_o$.

The symbols $ce_n(\vartheta)$ and $se_n(\vartheta)$ are fairly generally accepted for the odd and even Mathieufunctions. Not so, however, their normalization. In the sequel we will use McLachlan's normalization [2] and also his notations for the solutions of the modified Mathieu equation (3.3).

Using a familiar technique we substitute the trial solution (3.5) in (3.1) and obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} \varepsilon_n \left[ A_n'' - (a_n - 2h^2 \cosh 2\mu) A_n \right] ce_n(\vartheta) ce_n(\vartheta_o) + \\ &\quad + 2 \sum_{n=0}^{\infty} \left[ B_n'' - (b_n - 2h^2 \cos 2\vartheta) B_n \right] se_n(\vartheta) se_n(\vartheta_o) = \qquad (3.6) \\ &\qquad = -\delta(\mu - \mu_o)\, \delta(\vartheta - \vartheta_o), \end{aligned}$$

where $a_n$ is the characteristic number pertaining to $ce_n(\vartheta)$ and $b_n$ the characteristic number pertaining to $se_n(\vartheta)$. Multiplication of (3.6) with $ce_m(\vartheta)$ and subsequent integration of $\vartheta$ over the interval $(-\pi, \pi)$ gives

$$A_n'' - (a_n - 2h^2 \cosh 2\mu) A_n = -(2\pi)^{-1} \delta(\mu - \mu_o) \quad (3.7)$$

because of the orthonormal properties

$$\left.\begin{aligned} \varepsilon_n \int_{-\pi}^{\pi} ce_n(\vartheta) ce_m(\vartheta) d\vartheta = \varepsilon_n \int_{-\pi} se_n(\vartheta) se_m(\vartheta) d\vartheta &= \begin{cases} 0 & \text{for } n \neq m \\ 2\pi & \text{for } n = m \end{cases}, \\ \int_{-\pi}^{\pi} ce_n(\vartheta) se_m(\vartheta) d\vartheta = 0. & \end{aligned}\right\} \quad (3.8)$$

Also multiplication of (3.6) with $se_m(\vartheta)$ and subsequent integration of $\vartheta$ over the interval $(-\pi, \pi)$ gives by aid of (3.8)

$$B_n'' - (b_n - 2h^2 \cosh 2\mu) B_n = -(2\pi)^{-1} \delta(\mu - \mu_o). \quad (3.9)$$

Let $A_n^*(\mu)$ and $A_n^{**}(\mu)$ be two independent solutions of the homogeneous equation corresponding to (3.7) i.e. (3.3) with $\lambda = a_n$ and let $B_n^*(\mu)$ and $B_n^{**}(\mu)$ be two independent solutions of the homogeneous equation corresponding to (3.9) i.e. (3.3) with $\lambda = b_n$. The general solutions of (3.7) and (3.9) are then found to be

$$A_n(\mu) = \alpha_1 A_n^*(\mu) + \alpha_2 A_n^{**}(\mu) + \begin{cases} A_n(\mu)A_n^{**}(\mu_o)Wm_n/2\pi & \text{for } \mu > \mu_o \\ A_n(\mu_o)A_n^{**}(\mu)Wm_n/2\pi & \text{for } \mu < \mu_o \end{cases}, \tag{3.10}$$

and

$$B_n(\mu) = \beta_1 B_n^*(\mu) + \beta_2 B_n^{**}(\mu) + \begin{cases} B_n^*(\mu)B_n^{**}(\mu_o)Wn_n/2\pi & \text{for } \mu > \mu_o \\ B_n^*(\mu_o)B_n^{**}(\mu)Wn_n/2\pi & \text{for } \mu < \mu_o \end{cases}, \tag{3.11}$$

where the Wronskians are given by

$$Wm_n = A_n^*(\mu)A_n^{**\prime}(\mu) - A_n^{**}(\mu)A_n^{*\prime}(\mu), \tag{3.12}$$

$$Wn_n = B_n^*(\mu)B_n^{**\prime}(\mu) - B_n^{**}(\mu)B_n^{*\prime}(\mu). \tag{3.13}$$

In fact they are constants, which follows easily from the differential equation (3.3).

The conditions (3.2) for $G(\mu,\vartheta,\mu_o,\vartheta_o)$ will be satisfied if the functions $A_n(\mu)$ and $B_n(\mu)$ also satisfy these conditions. However, since (3.3) possesses no solutions which satisfy (3.2) both for $\mu \to \infty$ and for $\mu \to -\infty$, we must have $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$. Considering the remaining terms in (3.10) and (3.11) it is seen that the radiation condition (3.3) implies that

$$\left(\frac{d}{d\mu} - ihe^{\mu}\right)A_n^*(\mu) \to 0, \qquad \left(\frac{d}{d\mu} - ihe^{\mu}\right)B_n^*(\mu) \to 0 \quad \text{for } \mu \to \infty,$$

and

$$\left(\frac{d}{d\mu} - ihe^{\mu}\right)A_n^{**}(\mu) \to 0, \qquad \left(\frac{d}{d\mu} - ihe^{\mu}\right)B_n^{**}(\mu) \to 0 \quad \text{for } \mu \to -\infty.$$

These conditions are satisfied if we take

$$A_n^*(\mu) = Me_n^{(1)}(\mu), \qquad A_n^{**}(\mu) = Me_n^{(1)}(-\mu) \tag{3.14}$$

and

$$B_n^*(\mu) = Ne_n^{(1)}(\mu), \qquad B_n^{**}(\mu) = Ne_n^{(1)}(-\mu) \tag{3.15}$$

which follows from the asymptotic formulae given in the Appendix (A6). The Wronskians pertaining to the sets of solutions (3.14) and (3.15) have also been evaluated (A7). In the sequel we will omit the superscripts and write $Me_n$ and $Ne_n$ which is admissable in the present context since the corresponding functions of the second kind will not occur.

Collecting all results so far obtained we have for $\mu > \mu_o$:

$$G(\mu,\vartheta,\mu_o,\vartheta_o) = \frac{i}{4\pi}\sum_{n=0}^{\infty}\varepsilon_n\, Me_n(\mu)Me_n(-\mu_o)ce_n(\vartheta)ce_n(\vartheta_o)/Me_n(0)Fey_n'(0)$$
$$+\frac{i}{2\pi}\sum_{n=1}^{\infty}Ne_n(\mu)Ne_n(-\mu_o)se_n(\vartheta)se_n(\vartheta_o)/Ne_n'(0)Gey_n(0) \qquad (3.16)$$

and for $\mu < \mu_o$:

$$G(\mu,\vartheta,\mu_o,\vartheta_o) = \frac{i}{4\pi}\sum_{n=0}^{\infty}\varepsilon_n(Me_n(-\mu)Me_n(\mu_o)ce_n(\vartheta)ce_n(\vartheta_o)/Me_n(0)Fey_n'(0)$$
$$+\frac{i}{2\pi}\sum_{n=1}^{\infty}Ne_n(-\mu)Ne_n(\mu_o)se_n(\vartheta)se_n(\vartheta_o)/Ne_n'(0)Gey_n(0). \qquad (3.16$$

The functions $G_o, G_{Dr}, G_{Nr}$ etc. follow from (2.10), (2.12) - (2.15 by aid of (3.16) and (3.16'). Some simplification is obtained by usin (A7). We find for example

$$G_o = \frac{i}{2\pi}\sum_{n=0}^{\infty}\varepsilon_n Me_n(\mu)Ce_n(\mu_o)ce_n(\vartheta)ce_n(\vartheta_o)/Ce_n(0)Fey'(0)+$$
$$+\frac{i}{\pi}\sum_{n=1}^{\infty}Ne_n(\mu)Se_n(\mu_o)se_n(\vartheta)se_n(\vartheta_o)/Se_n'(0)Gey(0) \qquad (3.17)$$

for $\mu > \mu_o$ and

$$G_o = \frac{i}{2\pi}\sum_{n=0}^{\infty}\varepsilon_n\, Ce_n(\mu)Me_n(\mu_o)ce_n(\vartheta)ce_n(\vartheta_o)/Ce_n(0)Fey'(0)+$$
$$+\frac{i}{\pi}\sum_{n=1}^{\infty}\varepsilon_n Se_n(\mu)Ne_n(\mu_o)se_n(\vartheta)se_n(\vartheta_o)/Se_n'(0)Gey(0) \qquad (3.17')$$

for $\mu < \mu_o$.

Appendix

In this Appendix some formulae for the modified Mathieufunctions are collected. The numbers between braces refer to McLachlan's book

$$\left.\begin{aligned} Ce_n(z,h^2) &= ce_n(iz,h^2) && \{2.30\ (2),(3)\} \\ Se_n(z,h^2) &= -ise_n(iz,h^2) && \{2.30\ (4),(5)\} \end{aligned}\right\} \text{(A1)}$$

$$\left.\begin{aligned} Fe_n(z,h^2) &= C_n(h^2)z\ Ce_n(z,h^2)+F_n(z,h^2) && \{7.61\ (1),(2)\} \\ Ge_n(z,h^2) &= -S_n(h^2)z\ Se_n(z,h^2)+G_n(z,h^2) && \{7.61\ (3),(4)\} \end{aligned}\right\} \text{(A2)}$$

$$\left.\begin{aligned} F_{2n}(z,h^2) &= C_{2n}(h^2) \sum_{r=0}^{\infty} f_{2r+2}^{2n} \sinh(2r+2)z && \{7.61\ (1a)\} \\ F_{2n+1}(z,h^2) &= C_{2n+1}(h^2) \sum_{r=0}^{\infty} f_{2r+1}^{2n+1} \sinh(2r+1)z && \{7.61\ (2a)\} \\ G_{2n+1}(z,h^2) &= S_{2n+1}(h^2) \sum_{r=0}^{\infty} g_{2r+1}^{2n+1} \cosh(2r+1)z && \{7.61\ (3a)\} \\ G_{2n+2}(z,h^2) &= S_{2n+2}(h^2) \sum_{r=0}^{\infty} g_{2r}^{2n+2} \cosh 2rz && \{7.61\ (4a)\} \end{aligned}\right\} \text{(A3)}$$

For the meaning of the constants $C_n, S_n, f_r^n, g_r^n$ we refer to section 7.22 of McLachlan's book.

$$\left.\begin{aligned} Fey_n(z,h^2) &= \frac{Fey_n(0,h^2)}{Ce_n(0,h^2)} Ce_n(z,h^2) + \frac{Fey_n'(0,h^2)}{Fe_n'(0,h^2)} Fe_n(z,h^2) && \{13.21(4 \\ Gey_n(z,h^2) &= \frac{Gey_n'(0,h^2)}{Se_n'(0,h^2)} Se_n(z,h^2) + \frac{Gey_n(0,h^2)}{Ge_n(0,h^2)} Ge_n(z,h^2) && \{13.21(5 \end{aligned}\right. \text{(A4)}$$

$$\left.\begin{aligned} Me_n(z,h^2) &= Ce_n(z,h^2)+iFey_n(z,h^2) \\ Ne_n(z,h^2) &= Se_n(z,h^2)+iGey_n(z,h^2) \end{aligned}\right\} \text{(A5)}$$

In (A5) and after the superscript (1) of $Me_n$ and $Ne_n$ has been dropped. The corresponding functions of the second kind are of no interest in the present context.

$$\left.\begin{aligned} Me_{2n}(z,h^2) &\sim p_{2n}(h^2) \sqrt{2/\pi h}\ \exp\left[-\tfrac{1}{2}z+i(he^z-\tfrac{1}{4}\pi)\right] \\ Me_{2n+1}(z,h^2) &\sim p_{2n+1}(h^2) \sqrt{2/\pi h}\ \exp\left[-\tfrac{1}{2}z+i(he^z-\tfrac{3}{4}\pi)\right] \\ Ne_{2n+1}(z,h^2) &\sim s_{2n+1}(h^2) \sqrt{2/\pi h}\ \exp\left[-\tfrac{1}{2}z+i(he^z-\tfrac{3}{4}\pi)\right] \\ Ne_{2n+2}(z,h^2) &\sim s_{2n+2}(h^2) \sqrt{2/\pi h}\ \exp\left[-\tfrac{1}{2}z+i(he^z-\tfrac{1}{4}\pi)\right] \end{aligned}\right\} \text{(A6)}$$

For the constants $p_n$ and $s_n$ we refer to section 3 of Appendix 1 of

McLachlan's book.

From (A1),(A2) and (A3) it follows that $Ce_n(z,h^2)$ and $Ge_n(z,h^2)$ are even functions of z and that $Se_n(z,h^2)$ and $Fe_n(z,h^2)$ are odd functions of z. Taking this into account we can easily derive from (A4) and (A5) that

$$\left.\begin{aligned}
&Me_n(z,h^2)+Me_n(-z,h^2)=2Me_n(0,h^2)Ce_n(z,h^2)/Ce_n(0,h^2),\\
&Me_n(z,h^2)-Me_n(-z,h^2)=2i\ Fey_n'(0,h^2)Fe_n(z,h^2)/Fe_n'(0,h^2),\\
&Ne_n(z,h^2)+Ne_n(-z,h^2)=2i\ Gey_n(0,h^2)Ge_n(z,h^2)/Ge_n(0,h^2),\\
&Ne_n(z,h^2)-Ne_n(-z,h^2)=2\ Ne_n(0,h^2)Se_n(z,h^2)/Se_n'(0,h^2).
\end{aligned}\right\} \quad (A7)$$

The two Wronskians, defined by

$$Wm_n = Me_n(z,h^2)Me_n'(-z,h^2)-Me_n(-z,h^2)Me_n'(z,h^2),$$
$$Wn_n = Ne_n(z,h^2)Ne_n'(-z,h^2)-Ne_n(-z,h^2)Ne_n'(z,h^2)$$

can be easily shown to be constants. By means of (A7), or otherwise, their value is found to be

$$\left.\begin{aligned}
&Wm_n = -2i\ Me_n(0,h^2)Fey'(0,h^2),\\
&Wn_n = -2i\ Ne_n'(0,h^2)Gey(0,h^2).
\end{aligned}\right\} \quad (A8)$$

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